

Chapter 8: Dielectric Materials

*Definition of Electric flux Density, Polarization vector, Dielectric permittivity, and relative permittivity
Induced charge density
Boundary Conditions*

8.1 Introduction: In this chapter we will take a look at the second type of material, called insulator (with respect to circuits in ECE 140) or dielectrics (with respect to Electromagnetics). In an insulator, the electrons are tightly bound to the atom and cannot move over macroscopic distances under the influence of an applied field. Although, no material is a perfect insulator, for the time being we will assume that the motion of charges over macroscopic distances is negligible. When we apply an external electric field to an insulator, the electrons in the atom still feel the force and are displaced towards one side as compared to the nucleus as shown in the figure below.

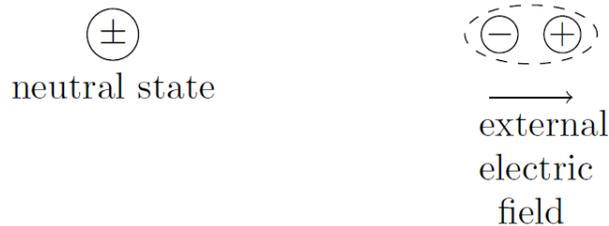


Figure 1. Dipole being induced in an atom due to an external electric field

The insulator is said to be “polarized” and when we consider the polarizability of the insulator, we refer to them as “Dielectrics”. The name “dielectric” was coined by Faraday (we will visit him again in the course) who extensively studied the material properties of dielectrics when placed between two electrodes. For students who want to go deep, an excellent resource is R. S. Elliott, *Electromagnetics*, IEEE Press., Piscataway, N J., 1993.

8.2 Polarization Vector

Polarization of dielectrics occurs due to the displacement of one or more electrons per atom over subatomic or microscopic distance as shown in Figure 1. The atom of the dielectric materials has their outermost electron shell nearly full resulting in the electrons becoming tightly bound to the atom. When a dielectric material is placed in an external electric field, the electrons feel a force and respond by shifting to one side as compared to the nuclei. The displacement, though very small as compared to conductors, nevertheless polarize the material and create electric dipoles. There will be an induced electric field inside the dielectric which will modify the total electric field in the system as shown in Figure 2. This will also induce charges on the surface and the volume of the material.

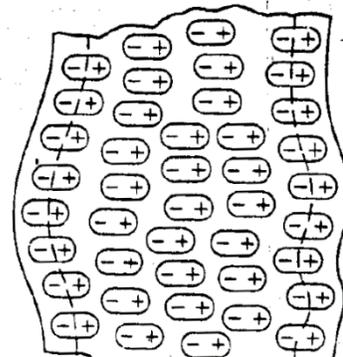


Figure 2. Dipoles polarized in the dielectric create induced charges on the surface and the volume of the material

As seen before, the dipole is characterized by a dipole moment, \vec{p} , given as $\vec{p} = q\vec{d}$ where q is the value of the charge and d is the distance between $-q$ and $+q$ charge and the direction of the vector is always from $-q$ charge to $+q$ charge. \vec{p} is a microscopic quantum mechanical property. In Engineering and classical physics, we are interested in knowing how the material responds as an ensemble to external electric fields. Any dimensions of interest will have lots of these dipoles. So we are interested in figuring out how the induced dipoles effects the material macroscopically. We define a macroscopic term called Polarization vector, P . Mathematically, it is defined as:

$$\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{i=0}^n \vec{p}_i}{\Delta v}$$

where n is the number of atoms per unit volume. The numerator represents the vector sum of the induced dipole moments contained in a very small volume Δv . The polarization vector, \vec{P} is a smoothed point function and is the volume density of the electric dipole moment. Without going into the derivations, the induced charge on the surface of the dielectric is given as:

$$\rho_s = \vec{P} \cdot \hat{n}$$

where \hat{n} is the normal unit vector to the surface pointing out of the dielectric. As we will see later, this is very important relationship and allows us to calculate the charge density induced on the surface of the conductors. The induced volume charge is given by the divergence of \vec{P} as:

$$\rho_v = -\nabla \cdot \vec{P}$$

Divergence of a vector field A at a point is the net outward flux of A per unit volume as the volume about the point tends to zero and is given as:

$$div \vec{A} = \lim_{\Delta v \rightarrow 0} \frac{\oint_S \vec{A} \cdot \vec{dS}}{\Delta v}$$

In Cartesian coordinates, the divergence is simply:

$$div A = \nabla \cdot A = \left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The physical significance of divergence is the rate at which density exits a given region of space e.g. you go on top of a hill and through water; how fast would water flow away from the point.

8.3 Electric Flux Density and Dielectric Constant

(the derivation will not be tested upon. However, it is important to see it to appreciate and understand what we are doing).

A polarized dielectric gives rise to a volume charge density, ρ_v . Thus, we expect the electric field intensity within the material to be different from that in free space. Let us see how we can include this effect. We have seen Gauss Law as:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

This is the integral form of Gauss Law and is applied in volume. What we know from the equation is that charge is a source (or sink if charge is negative) of E-field and the flux flows from the charge. So charge is a source and the electric field **diverges out of** (or into) the charge. We can also write the equation in point form as:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This is a point form of Gauss Law and operates at a point in space. As we have seen in conductors, the charge density ρ includes both the free charges (charge that can move freely in space) and the induced charges due the Electric fields created by the free charges.

In a dielectric material, the induced charges will be related to the polarization vector, \vec{P} . Thus, we have:

$$\nabla \cdot \vec{E} = \frac{\rho^{free} + \rho_v}{\epsilon_0} = \frac{\rho^{free} - \nabla \cdot \vec{P}}{\epsilon_0}$$

Rearranging we get:

$$\nabla \cdot \epsilon_0 \vec{E} + \nabla \cdot \vec{P} = \rho^{free}$$

Leading to:

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho^{free}$$

We define a new fundamental field quantity called electric flux density or electric flux displacement vector, \vec{D} given as: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$. The use of \vec{D} enables us to write the divergence relation between the electric field and the distribution of free charges in any medium without the necessity of dealing explicitly with the polarization vector \vec{P} or the polarization charge density. This is very powerful. **Unlike what we were doing in conductors, we don't have to worry about induced charges in our calculations!** The units of \vec{D} are C/m². Thus, the new Gauss Law in point form is:

$$\nabla \cdot \vec{D} = \rho^{free}$$

Converting it back into integral form we get the equation:

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{enc}^{free}$$

These are the 1st Maxwell's Equations (out of 4, so we are 25% of the way there). **Everything we talked about before for Gauss Law is applicable to this equation.** Except instead of E, we calculate D and the only enclosed charge we care about is the free charges; not induced ones.

When the dielectric properties of the medium are linear and isotropic (doesn't change with direction), the polarization vector will be directly proportional to the electric field intensity. We can write this proportionality constant as:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where χ_e is a dimensionless quantity called electric susceptibility. A dielectric medium is linear if χ_e is independent of \vec{E} and homogeneous if χ_e is independent of space co-ordinates. Substituting we get:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

We define $1 + \chi_e$ as **relative permittivity or dielectric constant**. We write:

$$\epsilon_r = 1 + \chi_e$$

ϵ_r is a dimensionless quantity. We thus, have:

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

The coefficient $\epsilon = \epsilon_0 \epsilon_r$ is the absolute permittivity with units of Farads/m (F/m). Thus, the relative permittivity ϵ_r measures how strongly does the material polarize with respect to free space. The value of ϵ_r for air is 1.00059 and thus, for most practical purposes, air is considered as free space.

By doing this math and defining \vec{D} , we have turned things around and made them quite simple. We first calculate \vec{D} using Gauss Law (for symmetric problems only) and only need to worry about the free charges (charges which have been placed in the system) and not have to consider the induced charged. Once we have that, we can calculate \vec{E} by simply dividing \vec{D} with the absolute permittivity of the material where we want to calculate the electric field. **Please note it is a local relation specific to a point. We only need to worry about the material property at the point we are calculating \vec{E} . Even if the material properties are varying in space, it does not matter to the relation at that point.** Once we have \vec{D} and \vec{E} , we can easily calculate P as

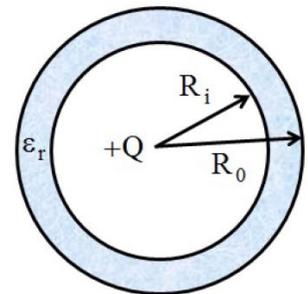
$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \epsilon_0 \epsilon_r \vec{E} - \epsilon_0 \vec{E} = \epsilon_0 (\epsilon_r - 1) \vec{E}$$

Once we have \vec{P} , we can get the polarized charge densities.

What is ϵ_r for a perfect conductor?

Example 1: A point charge is surrounded by a dielectric spherical shell of ϵ_r with an inner radius R_i and outer radius R_o . Calculate \vec{D} , \vec{E} , V and \vec{P} as a function of r.

Solution: We only have to worry about free charges which in this problem is Q at the center. So for \vec{D} the problem is only a point charge! We can start applying Gauss Law (the new form).



For $r < R_i$

$$D_I 4\pi r^2 = Q$$
$$D_I = \frac{Q}{4\pi r^2}$$

In this region, the material is air and thus, the absolute permittivity is simply ϵ_0 . Remember again \vec{E} is related to \vec{D} only locally, so only the absolute permittivity of the material where we are calculating \vec{E} matters. Thus, E_I in this region will be:

$$E_I = \frac{D_I}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}$$

There will be no polarization in this region and thus, $P = 0$.

For region $r > R_0$, we still have material with absolute permittivity ϵ_0 and the enclosed charge is still Q (again we are only looking at free charges). Thus, the functions for D , E and P are exactly the same as those in the previous region.

For the region $R_i < r < R_0$, the material has absolute permittivity $\epsilon_0\epsilon_r$ and the enclosed charge is still Q . Thus, D function still remains the same and will be given by:

$$D_{II} = \frac{Q}{4\pi r^2}$$

However, the Electric field will now be dependent on $\epsilon_0\epsilon_r$ and will be given by:

$$E_{II} = \frac{D_{II}}{\epsilon_0\epsilon_r} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}$$

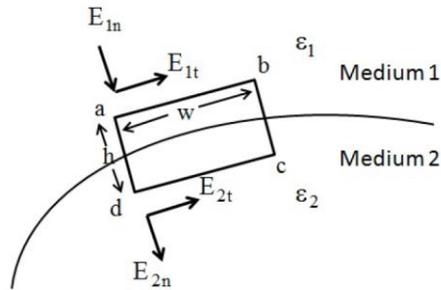
Polarization vector in the region will be given by:

$$P_{II} = \epsilon_0(\epsilon_r - 1)E = \frac{(\epsilon_r - 1)Q}{4\pi\epsilon_r r^2}$$

We now know E in space and potential, V , can be calculated from the values of E . Why don't you practice calculating V in different regions?

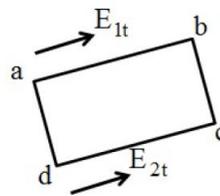
8.4 Boundary Conditions

Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between the two media. We want to understand how \vec{E} and \vec{D} fields change in crossing an interface. Consider the boundary between two dielectrics with absolute permittivities of ϵ_1 and ϵ_2 as shown in the figure below.



We want to calculate E_{2t} , E_{2n} given E_{1n} , E_{1t} as we go from medium 1 to medium 2.

Tangential Boundary condition: Let us first look at what happens to the tangential i.e. parallel field components. Like we did in conductors, let us make a loop around the boundary.



If we calculate the potential difference along the path starting from a and coming back to a, it should be 0. Thus,

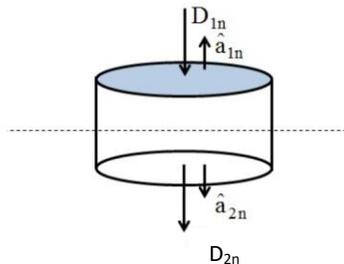
$$-\oint_{abcd} \vec{E} \cdot d\vec{l} = 0$$

Thus, $-E_{1t}w + E_{2t}w = 0$ which means that:

$$E_{1t} = E_{2t}$$

The tangential components of the electric fields have to be equal across the boundary.

Normal Boundary Conditions: To find the relation between the normal components of the field, let us apply Gauss Law across the boundary as shown in the figure below.



Applying Gauss Law (the new form) we get:

$$D_{2n}A - D_{1n}A = \rho_s^{free} A$$

where A is the cross sectional area of the pill box and ρ_s^{free} is any free charge density which may be on the boundary. Thus, we have

$$D_{2n} - D_{1n} = \rho_s^{free}$$

Remember this relationship is true as we have drawn in the figure and is specific to the directions in the figure.

If $\rho_s^{free} = 0$ i.e. the boundary has no free charge on it, then

$$D_{2n} = D_{1n}$$

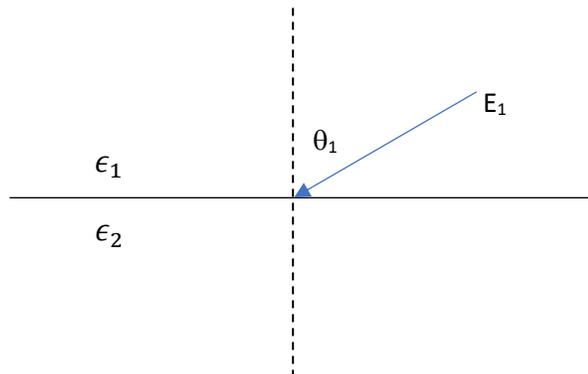
Which means $\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$. There is a discontinuity in the normal component of the Electric fields across the boundary. If $\epsilon_2 > \epsilon_1$, then $E_{1n} > E_{2n}$.

With these rules, we are now ready to tackle problems with dielectrics.

Example 2: We have a large planar boundary between two different dielectrics with absolute permittivity of ϵ_1 and ϵ_2 as shown in the Figure below. $\epsilon_2 > \epsilon_1$. Electric field is incident in dielectric 1 with amplitude E_1 and angle of θ_1 with the normal. There is no surface charge on the boundary. Answer the following questions:

(a) In dielectric 2, is angle θ_2 larger or smaller than θ_1 ?

(b) Calculate the value of E_2 and θ_2 .



Solution:

(a) E_1 will have a normal component, E_{1n} and a tangential component, E_{1t} as shown below. According to the boundary conditions, the tangential component, E_{2t} will be the same magnitude and the same length as E_{1t} . However, E_{2n} will be given as:

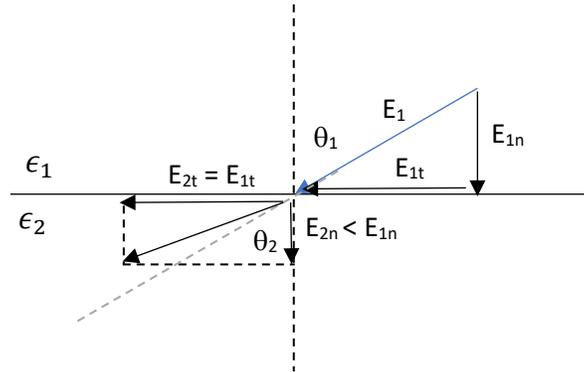
$$D_{1n} = D_{2n}$$

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

Therefore:

$$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n}$$

Since, $\epsilon_2 > \epsilon_1$, $E_{2n} < E_{1n}$. Let us visualize this in the vector diagram below.



As we can clearly see from the diagram, we expect $\theta_2 > \theta_1$.

(b) Let us know calculate the electric field values in the second dielectric using the boundary conditions. Using the diagram above we have:

$$E_{1n} = E_1 \cos(\theta_1)$$

$$E_{1t} = E_1 \sin(\theta_1)$$

As said before, the boundary conditions require:

$$E_{2t} = E_2 \sin(\theta_2) = E_{1t} = E_1 \sin(\theta_1) \quad (1)$$

And

$$E_{2n} = E_2 \cos(\theta_2) = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{\epsilon_1}{\epsilon_2} E_1 \cos(\theta_1) \quad (2)$$

To calculate the angle, θ_2 ; we can divide (2) by (1) and we get:

$$\tan(\theta_2) = \frac{\epsilon_2}{\epsilon_1} \tan(\theta_1)$$

Again, we can easily confirm that our conclusion in (a) was correct.

To calculate the value of E_2 , we can take the square of (1) and (2) and add them together (anytime you have sin and cos functions, this is the simplest way to remove the angle). We get:

$$(E_2 \cos(\theta_2))^2 + (E_2 \sin(\theta_2))^2 = \left(\frac{\epsilon_1}{\epsilon_2} E_1 \cos(\theta_1) \right)^2 + (E_1 \sin(\theta_1))^2$$

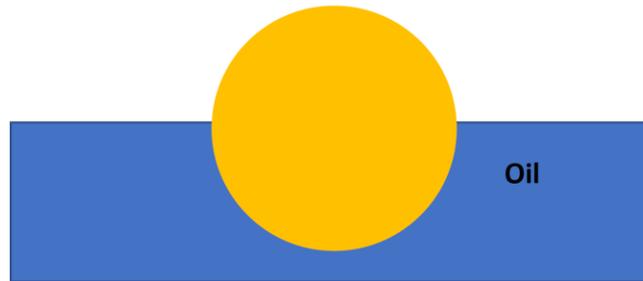
$$(E_2)^2 = (E_1)^2 \left(\left(\frac{\epsilon_1}{\epsilon_2} \cos(\theta_1) \right)^2 + (\sin(\theta_1))^2 \right)$$

$$E_2 = E_1 \sqrt{\left(\frac{\epsilon_1}{\epsilon_2} \cos(\theta_1) \right)^2 + (\sin(\theta_1))^2}$$

Thus, we have the values for the Electric field magnitude and the angle it makes with the normal.

Example 3. A solid gold ball is floated on an oil bath so that half the ball is above the oil and the other half immersed in oil. The oil has a relative permittivity of ϵ_r . Charge Q is released on the ball.

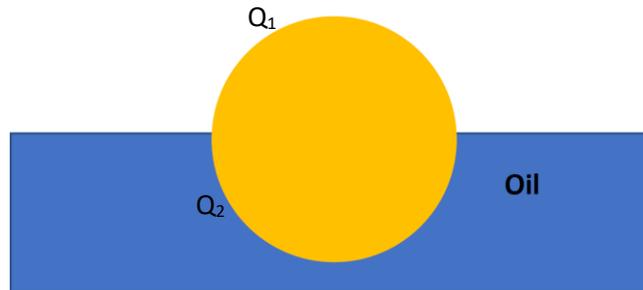
- Calculate the electric field in space.
- What is the charge density on the ball for the part above the oil and in the oil?
- What is the charge density in the oil on the boundary with the conductor?



Solution:

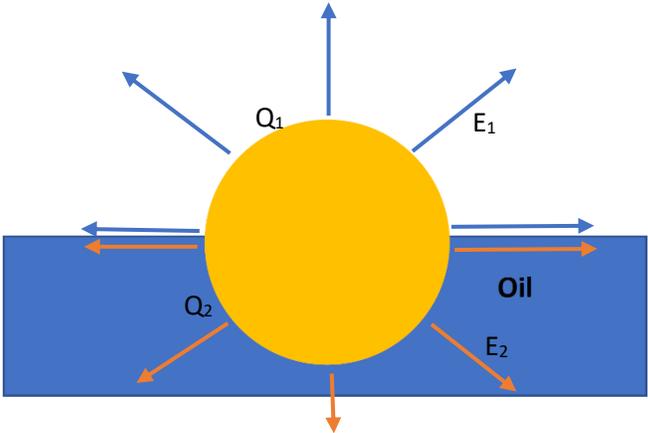
First question: We have released charge Q on the gold sphere. Obviously it is going to distribute itself on the surface of the sphere. Is it going to be uniformly distributed on the surface?

No! Oil can be polarized and hence, will attract more charge towards itself. Thus, the charge on the top part of the sphere and the bottom part of the sphere will have to be different as shown in the figure below.



Charge is still to be conserved and hence: $Q_1 + Q_2 = Q$. Charge is not distributed uniformly on the surface and it looks like we may not be able to use Gauss Law to calculate the Electric flux densities and from it the Electric fields.

We know one thing, that since the shape is a spherical conductor, Electric field should be radiating out. Let us call the field in air as E_1 and in oil as E_2 . Let us draw the field out and see what happens.



Let us go to the boundary between the oil and the air. E_1 is tangential to E_2 at the boundary. Thus, boundary conditions dictate that:

$$E_1 = E_2 = E$$

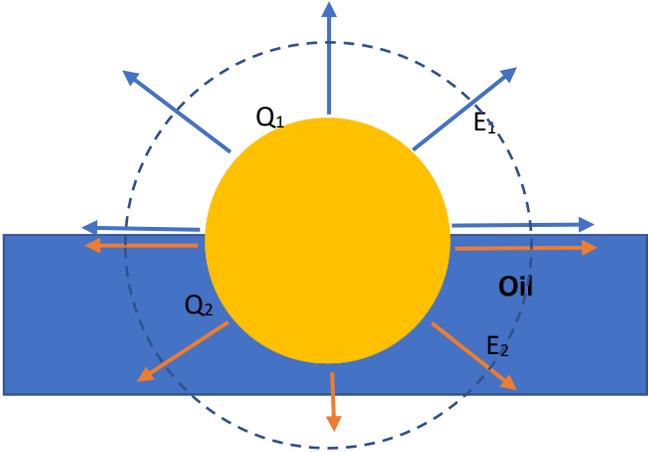
The electric flux density in the two regions are different and will be given in air as:

$$D_1 = \epsilon_0 E_1 = \epsilon_0 E$$

And in the oil as:

$$D_2 = \epsilon_0 \epsilon_r E_2 = \epsilon_0 \epsilon_r E$$

Let us apply Gauss Law on a Gaussian sphere of radius, r.



We get:

$$\oint \vec{D} \cdot \vec{dS} = Q_{free}^{enc}$$

The total charge enclosed is simply Q . The LHS of the equation needs to be split into the region with the air and the region with the oil as the values of the Electric flux density are different in these regions.

$$\int_{air} D_1 dS + \int_{oil} D_2 dS = Q$$

$$D_1 2\pi r^2 + D_2 2\pi r^2 = Q$$

Substituting the values of D_1 and D_2 we get a single equation in E :

$$\epsilon_0 E 2\pi r^2 + \epsilon_0 \epsilon_r E 2\pi r^2 = Q$$

Thus, the electric field at a distance “ r ” from the origin will be:

$$E = \frac{Q}{2\pi\epsilon_0(1 + \epsilon_r)r^2}$$

We can place the vector just by inspection and hence, the value of the Electric field will be:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0(1 + \epsilon_r)r^2} \hat{r}$$

Even though the charge is not distributed uniformly, boundary conditions have allowed us to calculate the Electric field using Gauss Law.

(b) Since, we have calculated the Electric field, the simplest way to calculate charge density on the conductor surface is to use conducting boundary conditions from Chapter 7. Electric field on the boundary of the conductor is related to the charge density on the conductor. For the sphere in air; the boundary conditions will dictate that:

$$E_1(r = R) = \frac{\rho_1}{\epsilon_0}$$

Therefore, the charge density on the conductor in air will be given as:

$$\rho_1 = \epsilon_0 E_1(r = R) = \epsilon_0 \frac{Q}{2\pi\epsilon_0(1 + \epsilon_r)R^2}$$

Similarly, for the conductor submerged in oil:

$$E_2(r = R) = \frac{\rho_2}{\epsilon_0 \epsilon_r}$$

Therefore, the charge density on the conductor in oil will be given as:

$$\rho_2 = \epsilon_0 \epsilon_r E_2(r = R) = \epsilon_0 \epsilon_r \frac{Q}{2\pi\epsilon_0(1 + \epsilon_r)R^2}$$

(c) Now we need to calculate the charge density induced in the oil, which is a dielectric. We need to calculate the Polarization vector, \vec{P} first.

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \epsilon_0(\epsilon_r - 1)\vec{E} = \frac{(\epsilon_r - 1)Q}{2\pi(1 + \epsilon_r)r^2}$$

Surface charge density at the surface of the dielectric is given as:

$$\rho_s = \vec{P} \cdot \hat{n}$$

Where \vec{P} will be the value at the boundary, $r = R$. Obviously, the charge will have to be negative. Thus, we can calculate the surface charge density as:

$$\rho_s = -\frac{(\epsilon_r - 1)Q}{2\pi(1 + \epsilon_r)R^2}$$

Unlike conductors where we had to think of induced charges; here we calculate D first, calculate E as a simple algebraic relationship. This allows us to calculate the polarization vector P and then we can calculate the charge density. All these are simple algebraic relations, and there in lies the beauty of defining D.